

CAPACITY ESTIMATES AND RIGIDITY OF DOMAINS WITH CORNERS

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ABSTRACT. We prove capacity inequalities involving the total mean curvature of hypersurfaces with boundary in convex cones and the mass of asymptotically flat manifolds with non-compact boundary. We then give the analogous of Pölya-Szegö, Alexandrov-Fenchel and Penrose type inequalities in this setting. Among the techniques used in this paper are the inverse mean curvature flow for hypersurfaces with boundary.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we aim to provide some new integral inequalities in terms of well known geometric quantities. Some of them are closely related to the isoperimetric problem in the theory of convex cones.

On the geometric side, we are interested in estimating the capacity of hypersurfaces in terms of the total mean curvature and the mass of asymptotically flat manifolds with non-compact boundary. We point out that there are still few examples where the exact value of the capacity of a set is known, thus estimates by using geometric terms are of great interest. The most known sharp capacity estimates were obtained by Szegö [31, 32], also including rigidity statements. From a physical point of view, the capacity of a compact set Ω in a Riemannian manifold M represents the electric charge flowing into $M \setminus \Omega$ through the boundary $\partial\Omega$ so that the electric potential of the field created by this charge is bounded by 1, see e.g. [25, § 2.1].

In part of this work, we focus on convex cones of \mathbb{R}^n and our motivation is due to the existence of a lot of interest in quantitative estimates for isoperimetric inequalities and existence of isoperimetric regions in such setting. Relevant contributions in this direction were obtained by several mathematicians including P. L. Lions, F. Pacella, J. Choe, F. Morgan, M. Ritoré, C. Rosales, A. Figali and E. Indrei (see, for instance, [19, 4, 24, 27, 28, 6] and references therein, as well as relevant discussion in [26, 17]).

Another important case treated here involves the mass of asymptotically flat manifolds with non-compact boundary, which is an invariant quantity of the asymptotic geometry. We make use of the mass to give upper bounds for the capacity. In fact, this is a natural question which was first studied by Bray in [2] and later by Bray and Miao [3], using the positive mass theorem and the monotonicity of the Hawking mass along the inverse mean curvature flow (IMCF), respectively. Recently, a positive mass theorem on manifolds admitting non-compact boundary

was settled by Almaraz, Barbosa and de Lima [1], by adapting the classical method established by Schoen and Yau [29, 30] and Witten [34].

Building on the ideas of Freire and Schwartz in their proof of mass-capacity inequalities [7], we obtain analogous results by considering now hypersurfaces with boundary evolving under inverse mean curvature flow, an approach introduced by Marquadt [21, 22], who constructed solutions by rewriting the flow as an equation for the level set of a function whose advantage is to allow “jumps” in a natural way. For different approaches of this geometric flow, we refer the reader to [15, 16, 22].

In what follows, let us consider the Euclidean cone $C = \vec{0} \ast U$ that is the union of geodesic segments from vertex $\vec{0}$ to the points of ∂U , where U is a domain of the sphere \mathbb{S}^{n-1} . We let α_{n-1} denote the volume of the unitary sector $C(\alpha, 1) := C \cap \{|x| \leq 1\}$ with solid angle α . Observe that if U is an open half-sphere the cone coincides with an open half-space.

Let $\Omega \subset C$ be a smooth domain. It is important to distinguish the boundary parts of Ω on ∂C and in the interior of C by writing

$$\partial_C \Omega := \overline{\partial \Omega \setminus \partial C} \quad \text{and} \quad \partial_{\partial C} \Omega := \partial \Omega \setminus \partial_C \Omega.$$

The Dirichlet energy of a map $\phi : C \setminus \Omega \rightarrow \mathbb{R}$ is defined as

$$E(\phi) := \frac{1}{(n-2)\alpha_{n-1}} \int_{C \setminus \Omega} |D\phi|^2 dv,$$

where dv is the volume element of $C \setminus \Omega$.

We are now ready to give the following definition of capacity.

Definition 1. Let C be a cone centered at the origin. Assume that $\Omega \subset C$ is a smooth bounded domain containing the vertex of the cone such that $\Sigma = \partial_C \Omega$ meets ∂C orthogonally. The *Capacity* of Σ is given by

$$\text{Cap}(\Sigma) = \inf E(\phi), \tag{1.1}$$

where the infimum is taken over all smooth functions $\phi : C \setminus \Omega \rightarrow \mathbb{R}$ with $\phi|_{\Sigma} = 0$ and approach to one at infinity.

We state a rigidity result for free boundary outer-minimizing mean convex domains in convex cones (a Pólya-Szegő type inequality as in [32, § 2(15)]). Convexity of the cone here means that the second fundamental form Π of $\partial C \setminus \vec{0}$ with respect to the outward unit normal μ is non-negative.

Theorem 1. *Let $C \subset \mathbb{R}^n$ be a smooth convex cone centered at the origin. Assume that Ω is a smooth bounded domain containing the vertex of C such that $\Sigma = \partial_C \Omega$ meets ∂C orthogonally and is strictly mean convex, i.e., $H > 0$. If one of the following hypotheses holds.*

- I) $n \leq 7$; or
- II) $n > 7$, Σ is free boundary outer-minimizing in $C \setminus \Omega$;

Then

$$\text{Cap}(\Sigma) \leq \frac{1}{(n-1)\alpha_{n-1}} \int_{\Sigma} H d\sigma.$$

Equality holds if and only if Ω is the intersection of a ball centered at the vertex of cone with C . In particular, Σ is a spherical cap.

Remark 1. The above result can be proved under the same hypothesis but with outer-minimizing replaced by star-shaped with respect to the center of C . In which case one can use the same approaching as in Theorem 2.

In the theory of convex bodies, Alexandrov-Fenchel inequalities to star-shaped Euclidean domains with mean convex boundary have recently generated a fair amount of interest, see e.g. [9] (see also [7], for the case of domains with outer-minimizing boundary). In this work, we are also able to prove estimations of the total mean curvature of a star-shaped hypersurfaces in convex cones. The precise statement is the following.

Theorem 2. *Let $C \subset \mathbb{R}^n$ be a smooth convex cone centered at the origin. Assume that Ω is a smooth bounded domain containing the vertex of C such that $\Sigma = \partial_C \Omega$ meets ∂C orthogonally and is strictly mean convex, i.e., then $H > 0$. If Σ is star-shaped in $C \setminus \Omega$, we have*

$$\frac{1}{2(n-1)\alpha_{n-1}} \int_{\Sigma} H d\sigma \geq \frac{1}{2} \left(\frac{\text{Area}(\Sigma)}{\alpha_{n-1}} \right)^{\frac{n-2}{n-1}}, \quad (1.2)$$

with equality achieved if and only if Ω is the intersection of a ball centered at the vertex of cone with C . In particular, Σ is a spherical cap.

Remark 2. It would be interesting to know if Theorem 2 can be generalized to k -convex starshaped domains in the sense of [9].

Next we give a version of the Poincaré–Faber–Szegő inequality relating the capacity of a region to its volume (the Euclidean corresponding result can be found in [31, § 2]).

Theorem 3. *Let $C \subset \mathbb{R}^n$ be a smooth convex cone centered at the origin. Assume that Ω is a smooth bounded domain containing the vertex of C such that $\Sigma = \partial_C \Omega$ meets ∂C orthogonally. If $C \setminus \Omega$ is connected, then*

$$\text{Cap}(\Sigma) \geq \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(C \cap \{|x| \leq 1\})} \right)^{\frac{n-2}{n}}, \quad (1.3)$$

with equality achieved if and only if Ω is the intersection of a ball centered at the vertex of cone with C . In particular, Σ is a spherical cap.

We say that the Riemannian manifold M is *cornered* (or *curve-faced polyhedral*) manifold of depth $d = 2$ if the boundary ∂M of M is decomposed into a union of faces $\Sigma_1 \cup \Sigma_2$, such that Σ_1 is transversal to Σ_2 and the boundary $\partial \Sigma_i$ of Σ_i is equal to $\Sigma_1 \cap \Sigma_2$ for $i = 1, 2$, see [10] for further discussion of cornered manifolds. Finally, we should mention that the Riemannian half Schwarzschild space (see Section 4 for definition) is a non-trivial example of cornered manifold of depth 2.

In order to state our next results more precisely, let us introduce some terminology. A cornered manifold (M^n, g) , $n \geq 3$, of depth 2 is said to be *conformally flat* if it is isometric to $(\mathbb{R}_+^n \setminus \Omega, u^{\frac{4}{n-2}} \delta)$, where $\Omega \subset \mathbb{R}_+^n$ is a smooth bounded set such that $\partial \Omega$ intersect $\partial \mathbb{R}_+^n$ orthogonally. Furthermore, assume that u normalized so that $u \rightarrow 1$ at ∞ . Let $\Sigma \cup S$ denote the boundary of M . We consider the space \mathcal{M} of all conformally flat metrics g on M such that:

- i) The scalar curvature of $g = u^{\frac{4}{n-2}}\delta$, denoted by R_g , is non-negative.
- ii) $\Sigma = \partial_{\mathbb{R}_+^n}\Omega$ and $S = \partial_{\partial\mathbb{R}_+^n}(\mathbb{R}_+^n \setminus \Omega)$ are mean convex and minimal with respect to the euclidean metric, respectively.
- iii) Σ is minimal and S is mean convex with respect to g .

The following theorem presents a mass-capacity inequality and a volumetric Penrose type inequality for conformally flat manifolds.

Theorem 4. *Let $\mathfrak{m}(g)$ be the mass of (M, g) . Assume that (M, g) is an asymptotically flat manifold with non-compact boundary such that $g \in \mathcal{M}$. If $u|_{\Sigma} \geq 2$, we have:*

$$\mathfrak{m}(g) \geq \text{Cap}(\Sigma, g), \quad (1.4)$$

$$\mathfrak{m}(g) \geq 2 \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(\mathbb{R}_+ \cap \{|x| \leq 1\})} \right)^{\frac{n-2}{n}}. \quad (1.5)$$

Equality holds in (1.4) or (1.5) if and only if g is the Riemannian half Schwarzschild metric.

Remark 3. We say that Let \underline{u} be the inner radius of a hypersurface Σ i.e. the infimum of the function u on Σ . We point out that, in fact, we can obtain the following inequalities:

$$\mathfrak{m}(g) \geq \frac{2\underline{u}}{2 + \underline{u}} \text{Cap}(\Sigma, g) \quad \text{and} \quad \mathfrak{m}(g) \geq \underline{u} \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(\mathbb{R}_+ \cap \{|x| \leq 1\})} \right)^{\frac{n-2}{n}}.$$

Remark 4. The spacetime Penrose inequality is a long-standing conjecture that has only been proved in a few cases. For instance, the Riemannian version in dimension three was proved by Huisken and Ilmanen [12] and by Bray [2]. It gives a relationship between the ADM mass of an end of the manifold and the area of each outermost minimal sphere bounding the end.

Remark 5. More details, examples and importance of cornered domains can be found in the work of Gromov [8], whose approach allow us to see closed manifolds as cornered manifolds of depth 0 and those with nonempty boundary as cornered manifolds of depth 1.

The paper is organized as follows. In Section 2, we give an overview about the IMCF for hypersurfaces with boundary. Then we provide an interplay between this geometric flow and the total mean curvature whose relationship plays a key role in this work. Afterwards, in Section 3, we relate the capacity and total mean curvature of hypersurfaces with boundary which meet a cone perpendicular. As a consequence we prove Theorem 1, 2 and 3. The last section is devoted to establish some definitions and gather results in order to prove Theorem 4.

2. TOTAL MEAN CURVATURE AND THE IMCF FOR HYPERSURFACES WITH BOUNDARY

In this section, we give a brief discussion of the inverse mean curvature flow for hypersurfaces that possesses boundary. Part of the proofs herein relies on modifications of the argument in [7] using the approach developed by Marquadt in [20, 21, 22].

Let Σ be a compact, smooth, orientable, manifold with compact, smooth boundary $\partial\Sigma$. Suppose that $X_0 : \Sigma \rightarrow M$ is a $C^{2,\alpha}$ -immersion such that $\Sigma_0 := X_0(\Sigma)$ has strictly positive mean curvature and is perpendicular to a fixed supporting $C^{2,\alpha}$ -hypersurface S without boundary in (M, g) , satisfying

$$X_0(\partial\Sigma) = X_0(\Sigma) \cap S \quad \text{and} \quad g(N_0, \mu \circ X_0) = 0 \quad \text{on } \Sigma,$$

where N_0 and μ are the unit normal vector fields on Σ and S , respectively. Let $X : \Sigma \times [0, T] \rightarrow M$ be a solution of IMCF for hypersurfaces with boundary

$$\begin{cases} \frac{\partial X}{\partial t} = \frac{1}{H}N & \text{in } \Sigma \times (0, T) \\ X(\partial\Sigma, t) = \Sigma_t \cap S, \quad g(N, \mu \circ X_0) & \text{on } \partial\Sigma \times (0, T) \\ X(\cdot, 0) = X_0, & \text{on } \Sigma, \end{cases} \quad (2.1)$$

where H , assumed to be positive, is the mean curvature of Σ in M with respect to N and $\Sigma_t = X(\Sigma, t)$. Since Σ is orthogonal to S , the outward unit co-normal ν of $\partial\Sigma$ coincides with μ along $\partial\Sigma$.

In particular when M is a convex cone and Σ is star-shaped with respect to the center of the cone, we have the following analogous statement to the one of Gerhard [8] for closed hypersurfaces.

Theorem 5 (Marquadt [21]). *Let $C \subset \mathbb{R}^n$ be a smooth convex cone centered at the origin. Let $X_0 : \Sigma \rightarrow \mathbb{R}^n$ such that $\Sigma_0 := X_0(\Sigma)$ is a compact $C^{2,\alpha}$ -hypersurface which is star-shaped with respect to the center of the cone and has strictly positive mean curvature. Furthermore, assume that Σ_0 meets ∂C orthogonally. Then there exists a unique embedding*

$$X \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Sigma \times [0, +\infty); \mathbb{R}^n) \cap C^\infty(\Sigma \times (0, +\infty); \mathbb{R}^n)$$

with $X(\partial\Sigma, t) \subset \partial C$ for all $t \geq 0$ satisfying (2.1). Furthermore, the rescaled embedding $X(\cdot, t)e^{t/(n-1)}$ converges smoothly to an embedding X_∞ , mapping Σ into a piece of a round sphere of radius $r_\infty = \left(\frac{\text{Area}(\Sigma_0)}{\text{Area}(\Sigma)}\right)^{1/(n-1)}$.

Now, we return to the general context. Let $\varphi : M \rightarrow \mathbb{R}^n$ be a function such that $\Sigma_t = \partial_M \Omega_t$ where $\Omega_t = \{\varphi < t\}$. As long as the mean curvature of Σ_t is strictly positive, the parabolic formulation (2.1) is equivalent to

$$\begin{cases} \operatorname{div}\left(\frac{D\varphi}{|D\varphi|}\right) = |D\varphi| & \text{in } M_0 := M \setminus \overline{\Omega_0} \\ D_\mu \varphi = 0 & \text{on } \partial M_0 := \partial_S M_0 \\ \varphi = 0 & \text{on } \partial_M \Omega_0. \end{cases} \quad (2.2)$$

The hypersurface flows in the outward normal direction with speed $N = \frac{1}{H}$ and it is easy to see that there is a problem if either $H = 0$ or H changes the sign on Σ_t .¹ To overcome these problems, Marquadt [20, 22] developed the notion of weak solutions for (2.2), proving the existence and uniqueness of such solutions guided by the ideas of Huisken and Ilmanen [12].

Consider a foliation $\{\Sigma_t\}$ defined by the level sets of the function given by weak solution of IMCF in $M \setminus \overline{\Omega_0}$. By Lemma 5.1 and 5.3 of [22], each $\Sigma_t := \partial_M \{\varphi < t\}$,

¹For instance, for non-star-shaped initial hypersurfaces, singularities may occur in finite time

$\varphi \in C_{loc}^{0,1}(M)$, is a $C^{1,\frac{1}{2}}$ -hypersurface up to a set of dimension less than or equal to $n - 8$ which possesses a weak mean curvature in L^∞ given by

$$\vec{H}(x) = |D\varphi(x)|N(x), \quad \text{where} \quad N(x) := \frac{D\varphi(x)}{|D\varphi(x)|}$$

for almost every $t > 0$, $x \in \Sigma_t$. Although the result originally stated in Lemma 5.1 of [22] does not includes any information about the regularity of the boundary, one can be extended up to the boundary applying [8]. Hence for those above values of t , Σ_t is orthogonal to S in the classical sense and any neighborhood of points $x \in S \cap \partial_M^* \Omega_t^2$

Definition 2. Let $\Sigma \subset M$ be a hypersurface that meets S orthogonally. Σ is called *free boundary outer-minimizing* if any other hypersurface $\hat{\Sigma}$ that is transversal to S and encloses Σ has

$$\text{Area}(\Sigma) \leq \text{Area}(\hat{\Sigma}).$$

We also say that Σ is free boundary strictly outer-minimizing if every hypersurface which encloses it and is transversal to S has strictly greater area.

Note that a free boundary outer-minimizing has $H \geq 0$, since otherwise there would exist an outward variation which would decrease its area. The following lemma gives the connection between outer-minimizing property and parabolic problem with Neumann boundary condition (2.2).

Lemma 6. *If φ is a solution of the equation (2.2), then Σ_t is free boundary outer-minimizing for all $t > 0$.*

Proof. Assume that Σ is any hypersurface enclosing Σ_t and let U be the region between Σ and Σ_t . Integrating by parts gives

$$\int_U \text{div} \left(\frac{D\varphi}{|D\varphi|} \right) dv = \int_\Sigma \left\langle N, \frac{D\varphi}{|D\varphi|} \right\rangle d\sigma - \text{Area}(\Sigma_t) + \int_{\partial_S U} \frac{1}{|D\varphi|} D_\mu \varphi dl.$$

Since the left hand side is equal to $\int_U |D\varphi| dv \geq 0$ and $D_\mu \varphi = 0$ on $\partial_S U$, we get

$$\text{Area}(\Sigma_t) \leq \text{Area}(\Sigma)$$

for all $t > 0$. □

Remark 6. Note that $\varphi \equiv t$ on U , when $\text{Area}(\Sigma) = \text{Area}(\Sigma_t)$.

Remark 7. In the two-dimensional case, the existence of free boundary outer minimizing sets follows from the Plateau's problem with partially free boundary, see for example [5].

In the next step we calculate the evolution of the total mean curvature under the flow (2.1). Before proceeding, we need the following technical lemma.

Lemma 7. *Let $C \subset \mathbb{R}^n$ be a smooth cone. Let $\Omega \subset C$ be a smooth bounded domain so that $\Sigma = \partial_C \Omega$ meets ∂C orthogonally. Consider a foliation $\{\Sigma_t\}_{t \geq 0}$ given by weak solution of IMCF in $C \setminus \Omega$, where $\Sigma_t = \partial_C \{\varphi \leq t\}$.*

² ∂^* represents the reduced boundary in the sense of the set of locally finite perimeter in \mathbb{R}^n .

If $\Phi \in C_c^{1,0}((0,t), \mathbb{R}_+^n)$, we have

$$-\int_{\Omega_t} \langle D\phi, N \rangle H dv \leq \frac{n-2}{n-1} \int_{\Omega_t} \phi H^2 dv - \int_{\partial_{\partial C} \Omega_t} \frac{\phi}{H} D_\mu H dl$$

where $\Omega_t = \{\varphi \leq t\}$ and $\phi = \Phi \circ \varphi : \Omega_t \rightarrow \mathbb{R}$.

Proof. First, one can argue as Lemma A.1 in [7] (even supposing φ a function of class C^3) to obtain the following integral expression:

$$\int_{\Sigma_t} \frac{1}{|D\varphi|} \langle N, DH \rangle d\sigma = \int_{\Sigma_t} \Delta_{\Sigma_t} (-|D\varphi|^{-1}) d\sigma - \int_{\Sigma_t} \frac{1}{|D\varphi|} |A|^2 d\sigma,$$

where $|A|$ is the square sum of the principal curvature of Σ_t . In the following the co-area formula, the divergence theorem and the fact that $H = |D\varphi|$ a.e. yield

$$\begin{aligned} \int_{\Omega_t} \phi \langle N, DH \rangle dv &= \int_0^t \Phi(s) \left(\int_{\Sigma_s} \frac{\langle N, DH \rangle}{|D\varphi|} d\sigma_s \right) ds \\ &\quad + \int_0^t \Phi(s) \left(\int_{\partial\Sigma_s} \frac{\langle N, DH \rangle}{|D\varphi|^2} D_\mu \varphi dl_s \right) ds \\ &= - \int_0^t \Phi(s) \int_{\partial\Sigma_s} \frac{D_\mu H}{H^2} dl_s ds - \int_{\Omega_t} \phi |A|^2 dv. \end{aligned} \quad (2.3)$$

On the other hand,

$$\operatorname{div} (\phi H N) = (\Phi' \circ \varphi) |D\varphi| H + \phi \langle N, DH \rangle + \phi H^2.$$

So, integrating over Ω_t and combining (2.3), we get

$$-\int_{\Omega_t} \langle D\phi, N \rangle H dv \leq \int_{\Omega_t} \phi (H^2 - |A|^2) dv - \int_{\partial_{\partial C} \Omega_t} \phi \frac{D_\mu H}{H} dl_t.$$

Denoting by $(\kappa_1, \dots, \kappa_{n-1})$ the principal curvature vector of Σ_t , we have by the Newton-MacLaurin's inequality for the expression $2 \sum_{i < j} \kappa_i \kappa_j$ that

$$H^2 - |A|^2 \leq \frac{n-2}{n-1} H^2,$$

with the equality holding only if Σ_t is umbilical.

In view of the above,

$$-\int_{\Omega_t} \langle D\phi, N \rangle H dv \leq \frac{n-2}{n-1} \int_{\Omega_t} \phi H^2 dv - \int_{\partial_{\partial C} \Omega_t} \frac{\phi}{H} D_\mu H dl_t.$$

□

We define the following quantity

$$\mathcal{I}(\Sigma) = \int_{\Sigma} H d\sigma. \quad (2.4)$$

Consider the foliation $\{\Sigma_t\}_{t \geq 0}$ defined by the level sets of the function given by weak solution of IMCF for hypersurfaces with boundary in $C \setminus \Omega$. We prove the following proposition about the functional \mathcal{I} .

Proposition 8. *We have*

$$\mathcal{I}(\Sigma_t) \leq \mathcal{I}(\Sigma) \exp \left[\left(\frac{n-2}{n-1} \right) \cdot t \right], \quad t \geq 0. \quad (2.5)$$

Proof. Using equality (3.3) of [20], we compute:

$$\frac{d}{dt} \mathcal{I}(\Sigma_t) = \int_{\Sigma_t} \left(H - \frac{|A|^2}{H} \right) d\sigma_t + \int_{\partial \Sigma_t} \frac{1}{H^2} D_\mu H dl_t.$$

Using once more the Newton-MacLaurin's inequality and the Neumann condition $D_\mu H = -H\Pi(N, N)$ derived in (3.6) of [22], we get

$$\frac{d}{dt} \mathcal{I}(\Sigma_t) \leq \frac{n-2}{n-1} \mathcal{I}(\Sigma_t) - \int_{\partial \Sigma_t} \frac{1}{H} \Pi(N, N) dl_t. \quad (2.6)$$

It remains to justify (2.6) even in the presence of jumps. In other words, when the flowing hypersurface jumps, the total mean curvature strictly decreases and one can to extend inequality (2.6) through many jump times.

Next consider the hypersurface $\Sigma' = \partial_C \{\varphi > 0\}$. From [22, §4], we know that Σ' is strictly area minimizing among hypersurfaces homologous to Σ . So define the following cutoff function $\Phi : [0, t] \rightarrow \mathbb{R}_+^n$ by

$$\Phi(s) = \begin{cases} 0 & \text{on } [0, \tau] \\ (s - \tau)/\varepsilon & \text{on } [\tau, \tau + \varepsilon] \\ 1 & \text{on } [\tau + \varepsilon, t - \varepsilon] \\ (t - s)/\varepsilon & \text{on } [t - \varepsilon, t]. \end{cases}$$

where $\tau \in (0, t)$ and $\varepsilon \in (0, \frac{t-\tau}{2})$. From Lemma 7, we have that $\phi = \Phi \circ \varphi$ satisfies

$$- \int_{\Omega_t} \langle D\phi, N \rangle H dv \leq \frac{n-2}{n-1} \int_{\Omega_t} \phi H^2 dv - \int_{\partial \partial_C \Omega_t} \frac{\phi}{H} D_\mu H dl. \quad (2.7)$$

On the other hand, making use of $D_\mu H = -H\Pi(N, N) \leq 0$, $H = |D\varphi|$ a.e. and the co-area formula we obtain the following inequality

$$\begin{aligned} \frac{1}{\varepsilon} \left[\int_{t-\varepsilon}^{\varepsilon} \mathcal{I}(\Sigma_s) ds - \int_{\tau}^{\tau+\varepsilon} \mathcal{I}(\Sigma_s) ds \right] &= - \int_0^t \Phi'(s) \mathcal{I}(\Sigma_s) ds \\ &= - \int_{\Omega_t} \langle D\phi, N \rangle H dv \\ &= \frac{n-2}{n-1} \int_{\Omega_t} \phi H^2 dv \leq \frac{n-2}{n-1} \int_0^t \Phi(s) \mathcal{I}(\Sigma_s) ds. \end{aligned}$$

Taking the limit as $\varepsilon \rightarrow 0$, we finally obtain

$$\mathcal{I}(\Sigma_t) \leq \mathcal{I}(\Sigma_\tau) + \frac{n-2}{n-1} \int_{\tau}^t \mathcal{I}(\Sigma_s) ds, \quad \text{for a.e. } 0 < \tau < t.$$

For $n \leq 7$, Lemma 5.1 and Lemma 5.5 of [22] imply that

$$\Sigma_{\tau_i} \rightarrow \Sigma' \quad \text{as } \tau_i \searrow 0,$$

locally in $C^{1,\beta}$, $\beta < \frac{1}{2}$. However, if $n > 7$ and Σ is free boundary outer-minimizing, then Σ coincides with Σ' and is disjoint from the singular set of φ , so the above convergence remains true. It follows from the first variation formula of area that

Σ_t possesses a weak mean curvature in L^1 (see (5.2) of [22]), which implies together the Riesz Representation Theorem that

$$\mathcal{I}(\Sigma_{\tau_i}) \rightarrow \mathcal{I}(\Sigma').$$

By the regularity result obtained in [33] (see also Theorem 1.3(iii) in [12]) and (see (1.15) in [12])

$$H_{\Sigma'} = 0 \text{ on } \Sigma' \setminus \Sigma \text{ and } H_{\Sigma'} = H_{\Sigma} \geq 0 \text{ a.e. on } \Sigma' \cap \Sigma,$$

we conclude that $\mathcal{I}(\Sigma') \leq \mathcal{I}(\Sigma)$. In particular,

$$\mathcal{I}(\Sigma_t) \leq \mathcal{I}(\Sigma) + \frac{n-2}{n-1} \int_0^t \mathcal{I}(\Sigma_s) ds. \text{ for a.e. } t > 0.$$

Therefore the proposition follows directly from Gronwall's Lemma. \square

3. CAPACITY INEQUALITIES

Our first result is a capacity inequality for certain hypersurfaces with boundary in convex cones. In the sequel, we proof Theorem 1, 2 and 3.

Proposition 9. *Let $C \subset \mathbb{R}^n$ be a smooth convex cone centered at the origin. Assume that Ω is a smooth bounded domain containing the vertex of C such that $\Sigma = \partial_C \Omega$ meets ∂C orthogonally and is strictly mean convex. If*

- $n \leq 7$; or
- Σ is free boundary outer-minimizing in $C \setminus \Omega$.

Then

$$\text{Cap}(\Sigma) \leq \frac{1}{(n-1)\alpha_{n-1}} \mathcal{I}(\Sigma).$$

Proof. We follow the ideas of Pólya and Szegő ([25, § 2], see also [3]). Let φ be a solution to the inverse mean curvature flow for hypersurfaces with boundary in $C \setminus \Omega$ with initial solution Σ . We set $\phi(x) = f \circ \varphi(x)$, where

$$f(t) = \Lambda \int_0^t \frac{1}{T(s)} ds, \quad \Lambda = \left(\int_0^\infty T(t)^{-1} dt \right)^{-1}.$$

Using the co-area formula we obtain

$$\begin{aligned} (n-2)\alpha_{n-1}\text{Cap}(\Sigma) &\leq \int_{C \setminus \Omega} |D\phi| dv \\ &= \int_0^\infty f'(t)^2 T(t) dt + \int_0^\infty f'(t) \int_{\partial \Sigma_t} D_\mu \varphi dl_t dt, \end{aligned} \tag{3.1}$$

where $T(t) = \int_{\Sigma_t} |D\varphi| d\sigma$.

From Proposition 8 and the fact that $D_\mu \varphi = 0$ on $\partial_{\partial C}(C \setminus \Omega)$, we have

$$\left(\int_0^\infty \frac{1}{T(t)} dt \right)^{-1} \leq \frac{n-2}{n-1} \int_\Sigma H d\sigma.$$

Therefore

$$\text{Cap}(\Sigma) \leq \frac{1}{(n-1)\alpha_{n-1}} \mathcal{I}(\Sigma).$$

\square

3.1. Proofs of Theorem 1, 2 and 3:

Proof of Theorem 1. By Proposition 9,

$$\text{Cap}(\Sigma) \leq \frac{1}{(n-1)\alpha_{n-1}} \mathcal{I}(\Sigma). \quad (3.2)$$

When the equality holds in (3.2), (2.5) is in fact an equality. Thus,

$$H^2 = (n-1)|A|^2 \text{ on } \Sigma_t, \text{ for a.e. } t \geq 0 \text{ and } \Pi(N, N) = 0,$$

So Σ_t is a union of pieces of totally umbilical spheres, for a.e. $t \geq 0$. Note that the evolution by IMCF for hypersurfaces with boundary does not have any jumps and the flow remains classical (otherwise (2.5) would be strictly). Another consequence is that Σ has to be connected.

In the following, this part of the proof follows ideas of Theorem 4.9 in [27]. Let $L(\vec{e})$ be the linear subspace generated by a vector $\vec{e} \in \mathbb{R}^n$. Given $x \in \Sigma$, the normal line $x + L(N(x))$ to Σ contains the center \mathbf{c} of a sphere containing Σ . Take a point $\tilde{x} \in \partial\Sigma$ at maximum distance from the vertex of the cone. As Σ meets ∂C orthogonally, we can assert that $N(\tilde{x})$ is proportional to \tilde{x} , and so $\mathbf{c} \in \partial C$.

In order to prove that $\mathbf{c} = \vec{0}$, and thus Σ is the intersection of a sphere centered at the vertex with the cone, we will argue by contradiction. Suppose that $\mathbf{c} \neq \vec{0}$. Now, pick a point $x \in \partial\Sigma \setminus \vec{0}$ and since $N(x)$ is tangent to ∂C (by the orthogonality condition), we have that $T_x \partial\Sigma$ contains the straight line $x + L(N(x))$. Hence \mathbf{c} and x belong to $\partial C \cap T_x(\partial C)$ and, by convexity, the segment line $l = \{\mathbf{c} + tx; t \in [0, 1]\}$ is contained in ∂C . Therefore $\partial S \subset T_{\mathbf{c}}(\partial C) \cap \partial C$ and thus ∂S is a great circle which bounds a flat region in $T_{x_0}(\partial C) \cap \partial C$, a contradiction. \square

Proof of Theorem 2. According to Marquadt [21], consider the foliation $\{\Sigma_t\}_{t \geq 0}$ given by IMCF in $C \setminus \Omega$ such that $\Sigma_0 = \Sigma$. Define the function

$$h(t) := \frac{\mathcal{I}(\Sigma_t)}{\text{Area}(\Sigma_t)^{\frac{n-2}{n-1}}}.$$

We recall that the area element evolves in the normal direction by

$$\frac{d}{dt} d\sigma_t = d\sigma_t.$$

Therefore the area satisfies

$$\frac{d}{dt} \text{Area}(\Sigma_t) = \text{Area}(\Sigma_t), \quad \text{Area}(\Sigma_t) = \exp(t) \text{Area}(\Sigma_0), \quad t \geq 0,$$

which together with Proposition 8 implies that $h(t)$ is non-increasing along IMCF in $C \setminus \Omega$. From Theorem 5, the problem (2.1) has a unique smooth solution for all time and the rescaled hypersurface Σ_t converges smoothly and exponentially to a unique piece of the round sphere as $t \rightarrow \infty$. Thus, at infinity $h(t)$ converges to $(n-1)\alpha_{n-1}^{1/(n-1)}$.

Therefore

$$\text{Area}(\Sigma)^{\frac{2-n}{n-1}} \mathcal{I}(\Sigma) = h(0) \leq h(t) \rightarrow (n-1)\alpha_{n-1}^{1/(n-1)}.$$

From this, (1.2) follows easily.

To prove the rigidity statement, we notice that if the inequality (1.2) becomes an equality we have $h(t) = h(0)$ for all t . Thus, (2.5) is also an equality and, by reasoning as in proof of Theorem 1, the rigidity follows. \square

First we recall that the infimum of (1.1) is attained by a unique solution ϕ ³ of the following mixed boundary value problem in $C \setminus \overline{\Omega}$:

$$\begin{cases} \Delta\phi = 0 & \text{in } C \setminus \overline{\Omega} \\ D_\mu\phi = 0 & \text{on } \partial_C(C \setminus \Omega) \\ \phi = 0 & \text{on } \partial_C\Omega \text{ and } \phi \rightarrow 1 \text{ as } |x| \rightarrow \infty, \end{cases} \quad (3.3)$$

where μ is the outward unit normal to ∂C . For a detailed proof of the existence and regularity of ϕ , see [13]⁴.

Inspired by the proof of Theorem 11 in [14], we prove

Proof of Theorem 3. Let φ be a function satisfying (3.3). Then

$$\text{Cap}(\Sigma) = \frac{1}{(n-2)\alpha_{n-1}} \int_{C \setminus \Omega} |D\varphi|^2 dv.$$

Since the level sets of φ give the foliation of $C \setminus \Omega$ we have by the co-area formula that

$$\int_{C \setminus \Omega} |D\varphi|^2 dv = \int_0^1 \int_{\Sigma_t} |D\varphi| d\sigma_t dt + \int_0^1 \left(\int_{\partial\Sigma_t} D_\mu\varphi dl_t \right) dt,$$

where $\Sigma_t = \varphi^{-1}(t)$ for $t \in [0, 1]$.

We observe that on the right hand side we have

$$\int_{C \setminus \Omega} |D\varphi|^2 dv \geq \int_0^1 \frac{\text{Area}(\Sigma_t)^2}{\int_{\Sigma_t} |D\varphi|^{-1} d\sigma_t} dt. \quad (3.4)$$

In addition, for our purposes it will be convenient to rewrite the integral on right hand side of (3.4) with help of the following expressions:

- $\frac{d}{dt} \text{Vol}(\Omega_t) = \int_{\Sigma_t} \frac{1}{|D\varphi|} d\sigma_t + \int_{\partial\Sigma_t} \frac{1}{|D\varphi|^2} D_\mu\varphi dL_t$
- $\text{Area}(\Sigma_t) \geq n \text{Vol}(C(\alpha, 1))^{\frac{1}{n}} \text{Vol}(\Omega_t)^{\frac{(n-1)}{n}}$,

where the second line is the isoperimetric inequality for convex cones proved by Lions and Pacella [17]).

Thus we obtain that

$$\text{Cap}(\Sigma) \geq \frac{1}{(n-2)\alpha_{n-1}} \int_0^1 \frac{\alpha_{n-1}^2 \left(\frac{\text{Vol}(\Omega_t)}{\text{Vol}(C(\alpha, 1))} \right)^{\frac{2(n-1)}{n}}}{d(\text{Vol}(\Omega_t))/dt} dt.$$

We let $R(t)$ denote be the radius of the sphere whose intersection with the cone has volume equal to $\text{Vol}(\Omega_t) = \text{Vol}(C(\alpha, 1))R(t)^n$. Thus we get

$$\text{Cap}(\Sigma) \geq \frac{1}{(n-2)} \int_0^1 \frac{R(t)^{n-1}}{R'(t)} dt. \quad (3.5)$$

Let $\hat{\Omega} \subset C$ be an open sector with radius $R(t)$, vertex at origin and same volume as Ω . Consider the function $\Psi : C \setminus \hat{\Omega} \rightarrow \mathbb{R}^n$ such that $\Psi^{-1}(t) = \hat{\Sigma}_t$, where

³The function ϕ is sometimes called of *electrostatic potential* of Σ .

⁴ In [13], the authors also provided a link between Neumann parabolicity and capacity of compact subsets.

$\hat{\Sigma}_t = \partial_C \hat{\Omega}_t$. Using that $\text{Area}(\hat{\Sigma}_t) = \alpha_{n-1} R(t)^{n-1}$ and $|D\Psi| = \frac{1}{R'(t)}$ on $\hat{\Sigma}_t$, we can rewrite (3.5) as

$$\text{Cap}(\Sigma) \geq \frac{1}{\alpha_{n-1}(n-2)} \int_0^1 \int_{\hat{\Sigma}_t} |D\Psi| d\hat{\sigma}_t dt.$$

Finally, using once more the co-area formula we deduce that

$$\text{Cap}(\Sigma) \geq \text{Cap}(\hat{\Sigma}) = \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(C(\alpha, 1))} \right)^{\frac{n-2}{n}}.$$

The rigidity statement is consequence of the one given by the isoperimetric inequality. \square

Remark 8. Using alternative arguments, an another proof of the above theorem can be given. The idea is as follow. Let C be a convex cone. Assume that $\Omega \subset C$ is a smooth domain containing the vertex of C , $\hat{\Omega}$ is a convex sector with $\text{Area}(\hat{\Omega}) = \text{Area}(\Omega)$ and u is a function in $V^p(\Omega) := \{v \in H^{1,p}(\Omega), v = 0 \text{ on } \Sigma = \partial_C \Omega\}$. One can define the *Symmetrization* of u as in [18, 19], which is a transformation associating u to a (unique) radial decreasing function $\hat{u} \in V^p(\hat{\Omega})$ having the same distribution function as u . So the proof would follow along the same line as that given by Lemma 14 and Lemma 15 of [7]. We also should mention that this symmetrization has the usual properties of the Schwarz symmetrization and can be used to prove classical isoperimetric inequalities for convex cones as in [17] or even to estimate the best Sobolev constant for embeddings [18].

4. MODEL CASE AND MASS-CAPACITY INEQUALITIES

We first give a brief discussion about asymptotic flatness and mass. A Riemannian manifold (M, g) , $n \geq 3$, with a non-compact boundary ∂M is *asymptotically flat* if there exists a compact $K \subset M$ and a diffeomorphism

$$\Psi : M \setminus K \rightarrow \mathbb{R}_+^n \cap \{|x| \leq 1\}$$

such that

$$g_{ij} = \delta_{ij} + O(|x|^{-p}),$$

and

$$\partial_k g_{ij} = O(|x|^{-p-1}) \quad \text{and} \quad \partial_t \partial_k g_{ij} = O(|x|^{-p-2})$$

for some $p > (n-2)/2$. The closed half-space \mathbb{R}_+^n endowed with the standard flat metric δ is example of metric asymptotically flat.

Assume that R_g and H_g are integrable in M and ∂M , respectively. In asymptotically flat coordinates, the mass of (M, g) is given by

$$\begin{aligned} \mathfrak{m}(g) := & \lim_{r \rightarrow +\infty} \left\{ \frac{1}{2(n-1)\omega_{n-1}} \sum_{i,j=1}^n \int_{\{x \in \mathbb{R}_+^n, |x|=r\}} (g_{ij,j} - g_{jj,i}) \frac{x_i}{r} d\sigma_r \right. \\ & \left. + \frac{1}{2(n-2)\omega_{n-1}} \sum_{\beta=1}^{n-1} \int_{\{x \in \partial \mathbb{R}_+^n, |x|=r\}} g_{\beta n} \frac{x_\beta}{r} d\sigma_r \right\}, \end{aligned} \quad (4.1)$$

where ω_{n-1} is the volume of the $n-1$ -dimensional sphere.

The above definition is independent of the particular choice of the chart at infinity what means that the mass is a geometric invariant. As already mentioned, we also have a Positive mass theorem in this context (see [1], Theorem 1.1), which states that an asymptotically flat manifold with non-negative scalar curvature and mean convex boundary has nonnegative mass if either $3 \leq n \leq 7$ or $n \geq 3$ and M is spin. Moreover the mass is zero if only if it is isometric to \mathbb{R}_+^n with the flat metric.

In the following, we recall the definition of the *half Schwarzschild space* of mass $\mathfrak{m} > 0$ which is the set $\{x \in \mathbb{R}_+^n; |x| \geq (\mathfrak{m}/2)^{\frac{1}{n-2}}\}$ endowed with the following conformal metric

$$g_{\mathfrak{m}} = \left(1 + \frac{\mathfrak{m}}{2}|x|^{2-n}\right)^{\frac{4}{n-2}} \delta, \quad \mathfrak{m} > 0.$$

This manifold is scalar-flat with a non-compact totally geodesic boundary $x_n = 0$ and the coordinate hemisphere of radius $(\mathfrak{m}/2)^{\frac{1}{n-2}}$ is the unique free boundary area-minimizing horizon. A straightforward computation gives that the mass $\mathfrak{m}(g_{\mathfrak{m}})$ is half the ADM mass of the standard Schwarzschild space. In fact, the double manifold of the half Schwarzschild space along its totally geodesic boundary is exactly the Schwarzschild space.

We start by recalling some well known formulae. If $g = u^{\frac{4}{n-2}} \delta$ for some positive smooth function u on M , we know that

$$\begin{cases} R_g = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{(n-2)} \Delta u + Ru \right) & \text{in } M \\ H_g = u^{-\frac{n}{n-2}} \left(\frac{2(n-1)}{(n-2)} D_\mu u + Hu \right) & \text{on } \partial M, \end{cases} \quad (4.2)$$

where μ and Δ denote the outward unit normal vector to ∂M and the Laplace-Beltrami operator with respect to δ , respectively.

As consequence of (4.2), those geometric assumptions on metrics in \mathcal{M} (i.e., scalar curvature $R_g \geq 0$ and mean curvature $H_g \geq 0$) are equivalent to assume that $\Delta u \leq 0$ in M and $D_\mu u \geq 0$ on S .

Another remarkable fact is that the maximum principle implies that a superharmonic function u on the half-space \mathbb{R}_+^n which is 1 at infinity and satisfies $D_{x_n} u = 0$ on $\partial \mathbb{R}_+^n$ is, in fact, identically 1. We emphasize that geometrically this means we can not conformally deform⁵ the half-space standard metric in a bounded region without decreasing the scalar curvature or mean curvature on the boundary somewhere.

Next we calculate the capacity of the Riemannian half Schwarzschild manifold. For $R > 0$, consider a function u on $\{x \in \mathbb{R}_+^n; |x| \geq R = (\mathfrak{m}/2)^{\frac{1}{n-2}}\}$ defined by

$$u = 1 - (R/|x|)^{n-2}.$$

We notice that u may be defined and harmonic in $\mathbb{R}_+^n \setminus \{0\}$ and satisfying $D_\mu u = 0$ on $\partial \mathbb{R}_+^n \setminus \{0\}$ as well.

Setting

$$\varphi = \frac{u}{1 + \left(\frac{R}{|x|}\right)^{n-2}},$$

⁵In fact, one can not have any compact deformation which is a consequence of the positive mass theorem

we obtain by conformal change that $\Delta_g \varphi = 0$, $D_{\mu_g} \varphi = 0$ on S . Note also that $\varphi = 0$ on Σ and $\varphi \rightarrow \infty$ at infinity. Thus, a straightforward calculation implies that $\text{Cap}(\Sigma, g_m) = \mathbf{m}$.

Assume that (M, g) is asymptotically flat satisfying $R_g \equiv 0 \equiv H_g$ near infinity, where $g = u^{\frac{4}{n-2}} \delta$. Thus,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^n \\ D_{x_n} u = 0 & \text{on } \partial \mathbb{R}_+^n, \end{cases}$$

for $|x|$ large which allows to write

$$u(x) = 1 - \frac{\mathbf{m}}{2} |x|^{2-n} + O(|x|^{1-n}). \quad (4.3)$$

Such expansion might simplifies some calculations. So the next step is to state an important approximation lemma by harmonically flat metric at infinity.

Lemma 10. (*Almaraz-Barbosa-de Lima [1]*) *Let (M, g) be an asymptotically flat manifold, conformally flat manifold with nonnegative scalar curvature $R_g \geq 0$ and mean convex boundary $H_g \geq 0$. For any $\epsilon > 0$ small enough there exists an asymptotically flat \bar{g} satisfying:*

- i) $R_{\bar{g}} \geq 0$ and $H_{\bar{g}} \geq 0$, with $R_{\bar{g}} \equiv 0$ and $H_{\bar{g}} \equiv 0$ near infinity;
- ii) \bar{g} is conformally flat near infinity;
- iii) $|\mathbf{m}(\bar{g}) - \mathbf{m}(g)| \leq \epsilon$.

Now we can relate the capacity defined by g and the euclidean capacity.

Proposition 11. *Let $g \in \mathcal{M}$. If (M, g) , $n \geq 3$, is asymptotically flat. Then*

$$\text{Cap}(\Sigma, g) \leq \text{Cap}(\Sigma) + \frac{\mathbf{m}(g)}{2}. \quad (4.4)$$

Equality holds if and only if $\Delta u = 0$ in M and $D_\mu u = 0$ on S .

Proof. Assume that near infinity u is harmonic and satisfies $D_\mu u = 0$ on S . Let us consider $w : \mathbb{R}_+^n \setminus \Omega \rightarrow (0, 1)$ satisfying the mixed boundary value problem (3.3) by replacing C by \mathbb{R}_+^n . Now, taking $\phi = \frac{w}{u}$ we have

$$\int_M |D_g \phi|_g^2 dv_g = \int_{\mathbb{R}_+^n \setminus \Omega} u^2 \left| D \left(\frac{w}{u} \right) \right|^2 dv = \lim_{\rho \rightarrow \infty} \int_{I(\rho)} u^2 \left| D \left(\frac{w}{u} \right) \right|^2 dv,$$

where $I(\rho) := \{x \in \mathbb{R}_+^n; |x| \leq \rho\} \setminus \Omega$. Note that

$$\begin{aligned} \int_{I(\rho)} u^2 \left| D \left(\frac{w}{u} \right) \right|^2 dv &= \int_{I(\rho)} |Dw|^2 - \left\langle D(w^2), \frac{Du}{u} \right\rangle dv \\ &+ \int_{I(\rho)} w^2 \frac{|Du|^2}{u^2} dv. \end{aligned}$$

Assume that ρ is sufficiently large. The divergence theorem and the fact that $w = 0$ on Σ yield

$$\begin{aligned} \int_{I(\rho)} \left\langle D(w^2), \frac{Du}{u} \right\rangle dv &= - \int_{I(\rho)} w^2 \operatorname{div} \left(\frac{Du}{u} \right) dv \\ &\quad + \int_{\mathbb{S}_+(\rho)} w^2 \frac{D_r u}{u} d\sigma_\rho + \int_{\partial_{\mathbb{R}_+^n} I(\rho)} w^2 \frac{D_\mu u}{u} d\sigma_\rho, \end{aligned}$$

where $r = |x|$ and $\mathbb{S}_+(\rho)$ is a large coordinate hemisphere of radius ρ . Since $\operatorname{div} \left(\frac{Du}{u} \right) = \frac{\Delta u}{u} - \frac{|D_\mu u|^2}{u}$, $\Delta u \leq 0$ in $\mathbb{R}_+^n \setminus \Omega$ and $D_\mu u \geq 0$ on $\partial_{\mathbb{R}_+^n}(\mathbb{R}_+^n \setminus \Omega)$ we conclude

$$\int_{I(\rho)} \left\langle D(w^2), \frac{Du}{u} \right\rangle dv \geq \int_{I(\rho)} w^2 \frac{|Du|^2}{u^2} dv + \int_{\mathbb{S}_+(\rho)} w^2 \frac{D_r u}{u} d\sigma_\rho. \quad (4.5)$$

Combining all the facts above and taking the limit as $\rho \rightarrow \infty$, we get

$$2 \int_{\mathbb{R}_+^n \setminus \Omega} u^2 \left| D \left(\frac{w}{u} \right) \right|^2 dv \leq (n-2) \omega_{n-1} \left(\operatorname{Cap}(\Sigma) + \frac{\mathbf{m}(g)}{2} \right),$$

and the inequality follows.

Assume that (4.4) is an equality, but either $\Delta u < 0$ in $\mathbb{R}_+^n \setminus \Omega$ or $D_\mu u > 0$ on $\partial_{\mathbb{R}_+^n}(\mathbb{R}_+^n \setminus \Omega)$. Therefore, suppose without loss of generality that there exists $x_0 \in \partial \mathbb{R}_+^n$ so that $D_\mu u \geq c > 0$ on $\partial_{\mathbb{R}_+^n}(\{|x - x_0| \leq b\})$ for some constants $b, c > 0$.

By Lemma 10, we can take a sequence of metrics $\{g_k\}$, $g_k = u_k \delta$, where each u_k approximates u . So, for $\rho, k \geq |x_0| + b$, we can rewrite (4.5) as

$$\begin{aligned} \int_{I(\rho)} \left\langle Dw^2, \frac{Du_k}{u_k} \right\rangle dv &\geq \int_{I(\rho)} w^2 \frac{|Du_k|^2}{u_k^2} dv + \int_{\mathbb{S}_+(\rho)} w^2 \frac{D_r u_k}{u_k} d\sigma_\rho \\ &\quad + \int_{I(\rho, x_0)} w^2 \frac{D_\mu u_k}{u_k} d\sigma_b \\ &\geq \int_{I(\rho)} w^2 \frac{|Du_k|^2}{u_k^2} dv + \int_{\mathbb{S}_+(\rho)} w^2 \frac{D_r u_k}{u_k} d\sigma_\rho + C_0 \end{aligned}$$

where $I(\rho, x_0) = \partial_{\mathbb{R}_+^n}(\{|x - x_0| \leq b\})$ and $C_0 > 0$ is a constant depending on w, u, c, b and n . Taking limits as $\rho \rightarrow \infty$ and $k \rightarrow \infty$ we get

$$\operatorname{Cap}(\Sigma, g) \leq \operatorname{Cap}(\Sigma) + \frac{\mathbf{m}(g)}{2} - 2\omega_{n-1}^{-1} C_0.$$

This leads to a contradiction because C_0 is positive.

The reciprocal is immediate because $\phi = \frac{w}{u}$ achieves the infimum for $\operatorname{Cap}(\Sigma, g)$. \square

We are ready to prove a result that gives lower bounds for the mass in terms of the total mean curvature.

Proposition 12. *Let $g \in \mathcal{M}$. If (M, g) , $n \geq 3$, is asymptotically flat. We have*

$$\mathbf{m}(g) \geq \frac{2u}{(n-1)\omega_{n-1}} \int_{\Sigma} H d\sigma, \quad (4.6)$$

where \underline{u} denotes the infimum of u on Σ . The equality holds if and only if $\Delta_g u = 0$ in M , $D_\mu u = 0$ on S and $u|_\Sigma$ is constant with $\underline{u} \geq 2$.

Proof. Assume that (M, g) is scalar flat and ∂M is minimal near infinity. From (4.2) and divergence theorem, we see that

$$\begin{aligned} \int_{I(\rho)} \Delta u dv &= \int_{\mathbb{R}_+^n \cap \{|x|=\rho\}} D_r u d\sigma_\rho - \int_\Sigma D_N u d\sigma - \int_{\partial_{\partial\mathbb{R}_+^n} I(\rho)} D_\mu u d\sigma \\ &= -\mathfrak{m}(g) \omega_{n-1} \frac{n-2}{4} + O\left(\frac{1}{\rho}\right) + \frac{n-2}{2(n-1)} \left(\int_\Sigma H u d\sigma + \int_{\partial_{\partial\mathbb{R}_+^n} I(\rho)} H u d\sigma \right). \end{aligned}$$

In the limit as $\rho \rightarrow \infty$ we see that

$$\begin{aligned} \mathfrak{m}(g) &= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}_+^n \setminus \Omega} R_g u^{-1} dv + \frac{2}{(n-1)\omega_{n-1}} \int_\Sigma H u d\sigma \\ &\quad + \frac{2}{(n-1)\omega_{n-1}} \int_{\partial_{\partial\mathbb{R}_+^n} (\mathbb{R}_+^n \setminus \Omega)} H u d\sigma. \end{aligned} \quad (4.7)$$

So the inequality (4.6) holds provided $R_g \geq 0$ and S is minimal.

Suppose that the equality holds in (4.6) and either $\Delta_g u < 0$ in M or $D_\mu u > 0$ on S somewhere. Take again a sequence of metrics $\{g_k\}$, $g_k = u_k \delta$, as in the proof of Proposition 11. Thus (4.7) becomes

$$\mathfrak{m}(g) < \frac{2}{(n-1)\omega_{n-1}} \int_\Sigma H u_k d\sigma,$$

taking the limit $k \rightarrow \infty$, this contradicts the fact that we are assuming the equality on (4.6).

It is easy to see that u is equals to its minimum on Σ , but it remains to show that $\underline{u} \geq 2$. Indeed, an analogous calculation using the divergence theorem gives

$$\begin{aligned} \int_{\mathbb{R}_+^n \setminus \Omega} u^2 R_g dv &= - \int_{\mathbb{R}_+^n \setminus \Omega} |Du|^2 dv - \omega_{n-1} (n-2) \frac{\mathfrak{m}(g)}{4} \\ &\quad + \frac{u^2 (n-2)}{2(n-1)} \left(\int_\Sigma H d\sigma + \int_{\partial_{\partial\mathbb{R}_+^n} (\mathbb{R}_+^n \setminus \Omega)} H d\sigma \right). \end{aligned}$$

Note that since

$$2 \int_{\mathbb{R}_+^n \setminus \Omega} |Du|^2 dv = (n-2) \omega_{n-1} (\underline{u} - 1)^2 \text{Cap}(\Sigma),$$

we can conclude using the equality in (4.6) that

$$(\underline{u} - 1) \text{Cap}(\Sigma) = \frac{\mathfrak{m}(g)}{2} \geq \frac{u}{2} \text{Cap}(\Sigma). \quad (4.8)$$

Therefore we have $\underline{u} \geq 2$. □

Putting the above inequalities we can prove the main result of this section.

Proof of Theorem 4. From Proposition 9, 11 and 12 it follows that

$$\text{Cap}(\Sigma, g) \leq \text{Cap}(\Sigma) + \frac{\mathfrak{m}(g)}{2} \leq \mathfrak{m}(g).$$

Therefore, we obtain (1.4).

The volumetric Penrose inequality (1.5) follows from Theorem 3, Proposition 9 and 12:

$$\frac{\mathfrak{m}(g)}{2} \geq \frac{2}{(n-1)\omega_{n-1}} \int_{\Sigma} H d\sigma \geq \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(\mathbb{R}_+^n \cap \{|x| \leq 1\})} \right)^{\frac{n-2}{n}}.$$

Now, suppose that (1.4) becomes equality. In particular (4.6) is also an equality and thus we can apply Proposition 12 to get

$$\begin{cases} \Delta_g u = 0 & \text{in } M \\ D_\mu u = 0 & \text{on } S \\ u \equiv 2 & \text{on } \Sigma. \end{cases} \quad (4.9)$$

On the other hand, according to Theorem 1, Σ is a hemisphere. This together with (4.9) imply that (M, g) is isometric to the Riemannian half Schwarzschild manifold.

Arguing similarly, if the equality occurs in (1.5) we also have that g is isometric to the Riemannian half Schwarzschild metric. □

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